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THE SAINT-VENANT PROBLEM AND PRINCIPLE IN ELASTICITY†

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Abstract—The traditional semi-inverse solution method of the Saint-Venant problem and the Saint-Venant principle, which were described in the Euclidian space under the Lagrange system formulation, are updated to be solved in the symplectic space under the conservative Hamiltonian system. Thus, the Saint-Venant problem and the Saint-Venant principle have been unified by the direct method. It is proved in the present paper that all the Saint-Venant solutions can be obtained directly via the zero eigenvalue solutions and all their Jordan normal forms of the corresponding Hamiltonian operator matrix and the Saint-Venant principle corresponds to neglect of all the non-zero eigenvalue solutions, in which the non-zero eigenvalue gives the decay rate. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The solution of elasticity for a prismatic domain is a classical problem which has lasted for more than a century. Because of the complication of its partial differential equations, the traditional method of separation of variables cannot be applied. Saint-Venant proposed the famous semi-inverse solution method (Saint-Venant, 1856a, b; Iesan, 1987), that some appropriate assumptions to the deformation should be made beforehand to find the solution, afterwards checking that the assumptions are valid. Thereafter, the semi-inverse solution became the classical solution method in elasticity (Sokolnikoff, 1965; Timoshenko, 1970). The boundary conditions at the two ends of the prismatic domain can only be satisfied in the sense of resultant equilibrium, termed the Saint-Venant principle. However, the semi-inverse method is not so ideal, as it can only find some solutions, but cannot assert that there has been no further solutions, nor how to find the remaining solutions, which satisfy the end conditions strictly.

At locations distant from the two ends of the cylinder, the solutions can be given approximately by those of Saint-Venant. As to the two end boundary conditions, the approximations are covered by the well-known Saint-Venant principle. The decay rates of stresses and displacement had been considered when self-equilibrated tractions are specified on the ends of the solid cylinder. Some solutions have been found only for a few cases where the geometry is of amenable mathematical form. Knowles and Horgan (1969) and Horgan (1974) established the H-K method, which applied energy-decay inequalities in investigating the decay of end effects in the isotropic and transversely isotropic cylinders. Klemm and Little (1970) employed the Love displacement solution and considered the case of the cylinder of solid circular section. Horgan and Knowles (1983) and Horgan (1989) gave reviews of the Saint-Venant principle, which concentrated on isotropic, anisotropic and composite structures. Robert and Keer (1987a, b) considered the axisymmetric and the asymmetric problem of the elastic circular cylinder using the biorthogonality relationship and the eigenfunction expansion technique. With the aid of the Papkovitch-Neuber solution, Stephen and Wang (1992) considered the self-equilibrated end load problem for a hollow circular cylinder. Stephen and Wang (1996) extended the application of Saint-Venant principle to frameworks and developed for determination of the decay eigenmodes of a continuum beam. However, these studies can not relate closely to the Saint-Venant solutions.

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Looking from the analogy theory between computational structural mechanics and optimal control (Zhong and Zhong, 1990, 1993), the Hamiltonian system theory can be introduced into the theory of elasticity (Zhong, 1991, 1994, 1995; Zhong and Yang, 1991); in these papers, the eigenfunction expansion method of the Hamiltonian operator matrix along the transverse cross section can be developed within the symplectic geometry space, thus the solution method reaches a new stage. Steele and Kim (1992) established a modified mixed variational principle for a class of problems with a spatial variable as the independent variable. Their result is a form exactly analogous to the classical mechanics of a dynamic system, with the equations of state exactly in the form of the canonical equations of Hamilton. Wang and Tang (1995) discussed a separable problem in elasticity with Hamiltonial system; however, the lateral boundary conditions have not been considered in accordance.

In this paper, based on the separation of variables of the Hamiltonian system, it can be proved that among its eigenfunction-vectors there exists an adjoint symplectic orthonormalisation relationship and the corresponding expansion theorem (Zhong, 1991, 1992, 1995). The eigenvalue zero and its corresponding eigenfunction-vectors in combination with their Jordan type eigenfunction-vectors play a specially important role. It will be shown in this paper that all the solutions of extensional, torsional and bending problems so far solved by the Saint-Venant semi-inverse method correspond to all the solutions of the multiple eigenvalue zero solutions of the Jordan type; and the end condition approximation, which is covered by the Saint-Venant principle, corresponds to neglect of all the non-zero eigenvalue solutions in the symplectic space under the Hamiltonian system. Thus, all the semi-inverse assumptions are relaxed, and not only the subspace spanned by all the Saint-Venant solutions is given its specific definite meaning, but also the exclusive argument can be shown that the eigenvalue zero corresponding to extended eigensolutions have no further solution, i.e., that the Saint-Venant semi-inverse method derived solutions have covered all the eigenvalue zero corresponding solutions and vice versa. Besides, the solutions in the neighbourhood of the ends, which are neglected in the Saint-Venant principle, can be obtained.

2. THE FUNDAMENTAL EQUATIONS

The homogeneous isotropic single connected cross section elastic body of prismatic domain is considered. The circular cylindrical coordinate (r, θ, z) is selected so that the z axis is along the longitudinal direction, and the original point locates at the central point of the cross section Ω. Ω is a singly connected domain, the outward normal $\mathbf{n}$ of its boundary contour $\partial\Omega$ has direction cosines (l, m). The boundary conditions at the contour $\partial\Omega$ are that this surface is free from traction

$$l\tau_{rz}+m\tau_{\theta z} = 0, \quad l\sigma_r+m\tau_{r\theta} = 0, \quad l\tau_{r\theta}+m\sigma_\theta = 0. \tag{1}$$

In this paper, the notations are the same as in Timoshenko (1970). The strain displacement relations can be described as

$$\varepsilon_r = \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \frac{1}{r}\left(\frac{\partial v}{\partial \theta}+u\right), \quad \gamma_{r\theta} = \frac{\partial v}{\partial r}+\frac{1}{r}\left(\frac{\partial u}{\partial \theta}-v\right),$$

$$\gamma_{rz} = \frac{\partial w}{\partial r}+\dot{u}, \quad \gamma_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta}+\dot{v}, \quad \varepsilon_z = \dot{w}, \tag{2}$$

where the dot represents differential with respect to z, namely $(\dot{\ }) = (\partial/\partial x)(\ )$. The z coordinate is analogous to the time coordinate in the dynamic problem; the stress-strain relation is given as

$$\sigma_r = \lambda(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)+2G\varepsilon_r = \lambda\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}+\dot{w}\right)+2G\frac{\partial u}{\partial r},$$

$$\sigma_\theta = \lambda(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)+2G\varepsilon_\theta = \lambda\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}+\dot{w}\right)+\frac{2G}{r}\left(\frac{\partial v}{\partial \theta}+u\right),$$

$$\sigma_z = \lambda(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)+2G\varepsilon_z = \lambda\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial \theta}+\frac{u}{r}+\dot{w}\right)+2G\dot{w},$$

$$\tau_{r\theta} = G\gamma_{r\theta} = G\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}-\frac{v}{r}\right), \tau_{rz} = G\gamma_{rz} = G\left(\dot{u}+\frac{\partial w}{\partial r}\right), \ \tau_{\theta z} = G\gamma_{\theta z} = G\left(\dot{v}+\frac{1}{r}\frac{\partial w}{\partial \theta}\right). \tag{3}$$

The equilibrium equations can be derived from the minimum potential energy variational principle,

$$\Pi = \int_0^L \iint_\Omega U r \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z, \quad \min_{u,v,w}\Pi, \tag{4}$$

where the potential energy density, U, can be expressed as

$$U = \tfrac{1}{2}[\lambda(\varepsilon_r+\varepsilon_\theta+\varepsilon_z)^2+2G(\varepsilon_r^2+\varepsilon_\theta^2+\varepsilon_z^2)+G(\gamma_{r\theta}^2+\gamma_{rz}^2+\gamma_{\theta z}^2)]. \tag{5}$$

The Lagrange function

$$\begin{aligned} L = rU = \frac{r}{2}\Bigg\{&\lambda\Bigg[\left(\frac{\partial u}{\partial r}\right)^2+\frac{u^2}{r^2}+\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2+\dot{w}^2+\frac{2u}{r}\frac{\partial u}{r}+\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial v}{\partial \theta} \\ &+2\dot{w}\frac{\partial u}{\partial r}+\frac{2u}{r^2}\frac{\partial v}{\partial \theta}+\frac{2u}{r}\dot{w}+\frac{2\dot{w}}{r}\frac{\partial v}{\partial \theta}\Bigg] \\ &+2G\left[\left(\frac{\partial u}{\partial r}\right)^2+\frac{u^2}{r^2}+\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta}\right)^2+\dot{w}^2+\frac{2u}{r^2}\frac{\partial v}{\partial \theta}\right]+G\left[\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2\right. \\ &+\left(\frac{\partial v}{\partial r}\right)^2+\frac{v^2}{r^2}+\frac{2}{r}\frac{\partial u}{\partial \theta}\frac{\partial v}{\partial r}-\frac{2v}{r^2}\frac{\partial u}{\partial \theta} \\ &\left.-\frac{2v}{r}\frac{\partial r}{\partial r}+\dot{v}^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2+\frac{2\dot{v}}{r}\frac{\partial w}{\partial \theta}+\left(\frac{\partial w}{\partial r}\right)^2+\dot{u}^2+2\dot{u}\frac{\partial w}{\partial r}\right]\Bigg\}, \end{aligned} \tag{6}$$

which implies that the displacement method is applied where the stresses are expressed with displacements. Equation (4) implies no external body force nor surface tractions at the lateral boundary, only at the two ends $z = 0$, L can there be external forces. It can be recognized that the displacement method has only one kind of variable and belongs to the Lagrange system approach, for which the displacement vector is

$$\mathbf{q} = \{u, v, w\}^{\mathrm{T}}, \tag{7}$$

from which the dual vector $\mathbf{p}$ is derived

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \begin{Bmatrix} rG\left(\dot{u} + \dfrac{\partial w}{\partial r}\right) \\ rG\left(\dot{v} + \dfrac{1}{r}\dfrac{\partial w}{\partial \theta}\right) \\ r\left[\lambda\left(\dot{w} + \dfrac{\partial u}{\partial r} + \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}\right) + 2G\dot{w}\right] \end{Bmatrix} = \begin{Bmatrix} r\tau_{rz} \\ r\tau_{\theta z} \\ r\sigma_z \end{Bmatrix}. \tag{8}$$

Based on the mutually dual vectors $\mathbf{q}$ and $\mathbf{p}$, the fundamental equations for the Hamiltonian system can be derived as follows.

$$H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}). \tag{9}$$

The variational eqn (4) turns to be

$$\delta \int_0^L \iint_\Omega [\mathbf{p}^{\mathrm{T}}\dot{\mathbf{q}} - H(\mathbf{q}, \mathbf{p})] \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z = 0. \tag{10}$$

From eqn (10) the dual eqns (11) and lateral traction free boundary conditions (1) can be obtained

$$\dot{\mathbf{q}} = \frac{\delta H}{\delta \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\delta \mathrm{H}}{\delta \mathbf{q}}. \tag{11}$$

Rewriting eqns (11) into matrix-vector form gives

$$\begin{Bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{B} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{p} \end{Bmatrix}, \tag{12}$$

where the operator submatrices are

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -\dfrac{\partial}{\partial r} \\ 0 & 0 & -\dfrac{\partial}{r\partial \theta} \\ -a_1\left(\dfrac{\partial}{\partial r} + \dfrac{1}{r}\right) & -a_1 \dfrac{\partial}{r\partial \theta} & 0 \end{bmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & a_1\left(\dfrac{\partial}{\partial r} + \dfrac{1}{r}\right) \\ 0 & 0 & a_1 \dfrac{\partial}{r\partial \theta} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{r\partial \theta} & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -a_4\left(\dfrac{r\partial^2}{\partial r^2} + \dfrac{\partial}{\partial r} - \dfrac{1}{r}\right) - G\dfrac{\partial^2}{r\partial \theta^2} & -a_5 \dfrac{\partial^2}{\partial r \partial \theta} + a_6 \dfrac{\partial}{r\partial \theta} & 0 \\ -a_5 \dfrac{\partial^2}{\partial r \partial \theta} - a_6 \dfrac{\partial}{r\partial \theta} & -G\left(\dfrac{r\partial^2}{\partial r^2} + \dfrac{\partial}{\partial r} - \dfrac{1}{r}\right) - a_4 \dfrac{\partial^2}{r\partial \theta^2} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} \frac{a_3}{r} & 0 & 0 \\ 0 & \frac{a_3}{r} & 0 \\ 0 & 0 & \frac{a_2}{r} \end{bmatrix}, \tag{13}$$

where $\mathbf{A}^{\mathrm{T}}$ is the adjoint operator of $\mathbf{A}$, and the negative signs come from the integration by parts (Zhong, 1991) in eqn (10), $a_1 = \lambda/(\lambda+2G)$, $a_2 = 1/(\lambda+2G)$, $a_3 = 1/G$, $a_4 = 4G(\lambda+G)/(\lambda+2G)$, $a_6 = G+a_4$, $a_5 = G+2\lambda G/(\lambda+2G)$. The eqns (12) should have corresponding boundary conditions for the lateral surface; they should be described by the dual vectors $\mathbf{q}$ and $\mathbf{p}$, as follows

$$l\left[a_4 r\frac{\partial u}{\partial r} + (a_5 - G)\left(\frac{\partial v}{\partial \theta} + u\right) + a_1 p_3\right] + mG\left(\frac{r\partial v}{\partial r} + \frac{\partial u}{\partial \theta} - v\right) = 0,$$

$$lG\left(\frac{r\partial v}{\partial r} + \frac{\partial u}{\partial \theta} - v\right) + m\left[a_4\left(\frac{\partial v}{\partial \theta} + u\right) + (a_5 - G)r\frac{\partial u}{\partial r} + a_1 p_3\right] = 0,$$

$$lp_1 + mp_2 = 0, \tag{14}$$

where u, v, w, p_1, p_2, p_3 are regarded as independent variables, from which all the eqns (12) and the lateral boundary conditions (14) have been expressed.

3. EIGENSOLUTION AND ADJOINT SYMPLECTIC ORTHO-NORMALIZATION

The solution of eqns (10) can make use of separation of variables. Let

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{B} & -\mathbf{A}^{\mathrm{T}} \end{bmatrix}, \quad \psi = \begin{Bmatrix} \mathbf{q} \\ \mathbf{p} \end{Bmatrix}. \tag{15}$$

Equation (12) is rewritten as

$$\dot{\psi} = \mathbf{H}\psi. \tag{16}$$

Therefore,

$$\psi = \psi_j(r, \theta)\mathrm{e}^{\mu_j z} \tag{17}$$

and the eigenequation is given as

$$\mathbf{H}\psi_j = \mu_j \psi_j, \tag{18}$$

where μ_j is the eigenvalue, $\mathbf{H}$ is a Hamiltonian operator matrix (Zhong, 1991, 1995). Introduce

$$\mathbf{J} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix}, \quad \mathbf{J}^{\mathrm{T}} = \mathbf{J}^{-1} = -\mathbf{J}, \quad \mathbf{J}^2 = \begin{bmatrix} -\mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}, \tag{19}$$

where $\mathbf{I}$ is the identity operator. To describe the behaviour of Hamiltonian operator matrix **H**, the boundary condition (14) should be considered simultaneously. Introduce the operation $\langle \bullet, \bullet, \bullet \rangle$ as

$$\langle \psi_1^{\mathrm{T}}, \mathbf{P}, \psi_2 \rangle = \int_{\Omega} \psi_1^{\mathrm{T}} \mathbf{P} \psi_2 \, \mathrm{d}r \, \mathrm{d}\theta, \tag{20}$$

where **P** is an arbitrary operator matrix. It can be verified by integration by parts and the boundary conditions that

$$\langle (\mathbf{J}\psi_1)^{\mathrm{T}}, \mathbf{H}, \psi_2 \rangle = \langle \psi_2^{\mathrm{T}}, \mathbf{H}^{\mathrm{T}}, (\mathbf{J}\psi_1) \rangle, \tag{21}$$

where ψ_1 and ψ_2 are arbitrary whole state vectors satisfying the boundary conditions, and $\mathbf{H}^{\mathrm{T}}$ is given as

$$\mathbf{H}^{\mathrm{T}} = \begin{bmatrix} \mathbf{F}^{\mathrm{T}} & -\mathbf{Q} \\ -\mathbf{G} & -\mathbf{F} \end{bmatrix} = \mathbf{JHJ}. \tag{22}$$

An operator matrix **H** satisfying eqn (22) is called the Hamiltonian by definition.

The eigenproblem (18) of a Hamiltonian operator can be proved that there are adjoint symplectic orthogonality relationships (Zhong, 1991, 1992, 1995; Zhong and Yang, 1991) between eigenvectors: if γ_j is an eigenvalue, $-\gamma_j$ must also be an eigenvalue. In other words, the eigensolution can be subdivided into the groups of (α) and (β):

$$(\alpha)\ \mu_{\alpha j}, \quad \mathrm{Re}(\mu_{\alpha j}) > 0 \quad \text{or} \quad \mathrm{Im}(\mu_{\alpha j}) > 0 \quad \text{when} \quad \mathrm{Re}(\mu_{\alpha j}) = 0 \ (j = 1,2,3\ldots)$$

$$(\beta)\ \mu_{\beta j}, \quad \mu_{\beta j} = -\mu_{\alpha j} \quad (j = 1,2,3\ldots) \tag{23}$$

and the corresponding eigenfunction-vectors can be denoted as $\psi_{\alpha j}$ and $\psi_{\beta j}$, respectively. Between any two of them, there are adjoint symplectic ortho-normalization relationships

$$\langle \psi_{\alpha_i}^{\mathrm{T}}, \mathbf{J}, \psi_{\beta_j} \rangle = \delta_{i_j}, \quad \langle \psi_{\beta_i}^{\mathrm{T}}, \mathbf{J}, \psi_{\alpha_j} \rangle = -\delta_{i_j}, \quad \langle \psi_{\alpha_i}^{\mathrm{T}}, \mathbf{J}, \psi_{\alpha_j} \rangle = \langle \psi_{\beta_i}^{\mathrm{T}}, \mathbf{J}, \psi_{\beta_j} \rangle = 0. \tag{24}$$

According to the characteristics of a Hamiltonian system, an arbitrary whole state vector ψ can always be expanded by the linear combination of the eigenfunction-vectors as

$$\psi = \sum_{i=1}^{\infty} (\alpha_i \psi_{\alpha_i} + \beta_i \psi_{\beta_i}), \tag{25}$$

where ψ_{α_i}, ψ_{β_i} are functions of r and θ, and coefficients α_i, β_i are functions of z. Using the adjoint symplectic ortho-normalization relationships gives

$$\alpha_i = -\langle \psi_{\beta_i}^{\mathrm{T}}, \mathbf{J}, \psi \rangle, \quad \beta_i = \langle \psi_{\alpha_i}^{\mathrm{T}}, \mathbf{J}, \psi \rangle. \tag{26}$$

Substituting eqn (25) into eqn (16) and using eqn (18) gives

$$\dot{\alpha}_i = \mu_{\alpha i} \alpha_i, \quad \dot{\beta}_i = -\mu_{\alpha i} \beta_i. \tag{27}$$

Hence (writing μ_i instead of $\mu_{\alpha i}$),

$$\alpha_i = \alpha_{i0} e^{\mu_i z}, \quad \beta_i = \beta_{i0} \mathrm{e}^{-\mu_i z}, \tag{28}$$

where the integration constants α_{i0} and β_{i0} should be solved by the boundary conditions.

4. THE ZERO EIGENVALUE SOLUTIONS

Because of traction free lateral boundary conditions, there must be eigenvalue zero solutions, and it may have various orders of principal vectors. When the end conditions are given, the solutions are unique. All the semi-inverse method solutions of Saint-Venant type, the extensional, torsional and bending solutions can be derived directly from the Jordan subsidiary eigenvectors of various orders with no assumption. The detailed derivations are given below.

4.1. *The eigensolutions of zero eigenvalue*

Let $\mu = 0$ in eqn (18), all the 4 fundamental eigensolutions of the eigenequation $\mathbf{H}\psi = 0$ are

$$\begin{aligned}
\psi_1^{(0)} &= \{u_1^{(0)} = \sin\theta,\quad v_1^{(0)} = \cos\theta,\quad w_1^{(0)} = 0;\quad p_{11}^{(0)} = 0;\quad p_{21}^{(0)} = 0;\quad p_{31}^{(0)} = 0\}^{\mathrm{T}},\\
\psi_2^{(0)} &= \{-\cos\theta,\quad \sin\theta,\quad 0;\quad 0;\quad 0;\quad 0\}^{\mathrm{T}},\\
\psi_3^{(0)} &= \{0;\quad 0;\quad 1;\quad 0;\quad 0;\quad 0\}^{\mathrm{T}},\\
\psi_4^{(0)} &= \{0;\quad r;\quad 0;\quad 0;\quad 0;\quad 0\}^{\mathrm{T}}.
\end{aligned} \tag{29}$$

The geometrical interpretation of these solutions are the translations along three directions and rotation of the cylinder.

4.2. *The solutions of Jordan normal form of first order*

The solutions of Jordan normal form are originally from matrix algebra; the same story can also be applied to the eigensolutions of partial differential equations.

The governing equations for finding the first order subsidiary eigensolutions of Jordan form are

$$\mathbf{H}\psi_i^{(1)} = \psi_i^{(0)} \quad (i = 1, 2, 3, 4). \tag{30}$$

For the cases of $i = 1, 2$, it is easy to find the solutions $\psi_i^{(1)}$ for $(i = 1, 2)$, that

$$\begin{aligned}
\psi_1^{(1)} &= \{0,\ 0,\quad -r\sin\theta,\quad 0,\quad 0,\quad 0\}^{\mathrm{T}},\\
\psi_2^{(1)} &= \{0,\ 0,\quad r\cos\theta,\quad 0,\quad 0,\quad 0\}^{\mathrm{T}}.
\end{aligned} \tag{31}$$

The fact should be mentioned that the above solutions can arbitrarily superimpose any linear combination of eqns (29), but it can only cause trivial repetition and is hence cancelled. The subsidiary eigensolutions of eqn (31), $\psi_1^{(1)}$, $\psi_2^{(1)}$, are not directly solutions of the original problem; however, the corresponding solutions of the original problem can be composed as

$$\psi = \{\mathbf{q}^{\mathrm{T}}, \mathbf{p}^{\mathrm{T}}\}^{\mathrm{T}} = \psi_1^{(1)} + z\psi_1^{(0)} = \{z\sin\theta,\quad z\cos\theta, -r\sin\theta,\quad 0,\quad 0,\ 0\}^{\mathrm{T}}, \tag{32}$$

$$\psi = \psi_2^{(1)} + z\psi_2^{(0)} = \{-z\cos\theta,\quad z\sin\theta, r\cos\theta,\quad 0,\quad 0,\ 0\}^{\mathrm{T}}. \tag{33}$$

Evidently, the solutions (32) and (33) are the rigid body rotation. The six solutions above are all rigid body motions. It gives the method of making use of the subsidiary eigenvectors.

For the case of $i = 3$, solving eqns (30) gives the subsidiary eigensolution

$$\psi_3^{(1)} = \{-\nu r,\quad 0,\quad 0,\quad 0,\quad 0,\quad Er\}^{\mathrm{T}}, \tag{34}$$

where $\nu = \lambda/[2(\lambda+G)]$, the Poisson's ratio, $E = 2G+G\lambda/(\lambda+G)$, the Young's modulus. The solution of the original problem can be composed in the same way

$$\psi = \psi_3^{(1)}+z\psi_3^{(0)} = \{-\nu r, 0, z, 0, 0, Er\}^{\mathrm{T}}. \tag{35}$$

Its physical meaning is the simple extension solution.

For $i = 4$, the subsidiary eigensolution is

$$\psi_4^{(1)} = \left\{0, \quad 0, \quad \varphi, \quad Gr\frac{\partial\varphi}{\partial r}, \quad Gr\left(\frac{\partial\varphi}{\partial\theta}+r^2\right), \quad 0\right\}^{\mathrm{T}}. \tag{36}$$

The solution can be described as

$$\psi = \psi_4^{(1)}+z\psi_4^{(0)} = \left\{0, \quad rz, \quad \varphi, \quad Gr\frac{\partial\varphi}{\partial r}, \quad G\left(\frac{\partial\varphi}{\partial\theta}+r^2\right), \quad 0\right\}^{\mathrm{T}}, \tag{37}$$

where the function φ is the solution of the following Neumann problem

$$\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{\partial\theta^2}\right)\varphi = 0, \quad \left.\frac{\mathrm{d}\varphi}{\mathrm{d}n}\right|_{\partial\Omega} = -mr. \tag{38}$$

Equation (36) is the Saint-Venant free torsional semi-inverse solution. It is easy to verify that

$$\int_{\partial\Omega}\frac{\mathrm{d}\varphi}{\mathrm{d}n}\mathrm{d}s = 0.$$

hence the solution of eqns (38) does exist.

4.3. *The solutions of Jordan normal form of second order*

The above subsection has given all the first order Jordan form subsidiary eigensolutions, thus the next layer Jordan form subsidiary solution must be considered. The respective equation is

$$\mathbf{H}\psi_i^{(2)} = \psi_1^{(1)} \quad (i = 1,2,3,4). \tag{39}$$

Let us solve them successively. For $i = 1$, consider eqn (39) and the boundary conditions (14); their solutions can easily be found as

$$\psi_1^{(2)} = \{\nu r^2\cos\theta/2, \quad \nu r^2\sin\theta/2, 0, 0, 0, -Er^2\cos\theta\}^{\mathrm{T}}, \tag{40}$$

$$\psi_2^{(2)} = \{\nu r^2\sin\theta/2, \quad -\nu r^2\cos\theta/2, 0, 0, 0, -Er^2\sin\theta\}^{\mathrm{T}}. \tag{41}$$

The vectors $\psi_1^{(2)}$ and $\psi_2^{(2)}$ themselves are not solutions of the original problem; however, from them the solutions can be composed as

$$\psi = \psi_i^{(2)}+z\psi_i^{(1)}+\frac{z^2}{2}\psi_i^{(0)} \quad (i = 1,2), \tag{42}$$

the physical interpretation of which is the pure bending in two planes. The only stress σ_z of pure bending is linearly distributed along the cross section. Hence, with no loss of generality, the coordinate axis can be arranged as the central principal inertia axes of the cross section.

For $i = 3$, 4, it can be proved that there is no solution satisfying the equations and the lateral boundary conditions (14), hence the corresponding chains of the Jordan normal form break here.

4.4. *The solutions of Jordan normal form of third order*

At the third level of Jordan normal form, only $i = 1, 2$ can be considered. The equations are

$$\mathbf{H}\psi_i^{(3)} = \psi_i^{(2)} \quad (i = 1, 2). \tag{43}$$

The solutions of the above equations should be found as

$$\begin{aligned}
\psi_1^{(3)} = \Big\{&0, \quad 0, \quad \varphi_1 + r^2\cos\theta/4, \quad G\left[r\frac{\partial\varphi_1}{\partial r} + (3+2\nu)r^3\cos\theta/4\right], \\
&G\left[\frac{\partial\varphi_1}{\partial\theta} - (1-2\nu)r^3\sin\theta/4\right], \quad 0\Big\}^{\mathrm{T}}, \\
\psi_2^{(3)} = \Big\{&0, \quad 0, \quad \varphi_2 + r^2\sin\theta/4, \quad G\left[r\frac{\partial\varphi_2}{\partial r} + (3+2\nu)r^3\sin\theta/4\right], \\
&G\left[\frac{\partial\varphi_2}{\partial\theta} + (1-2\nu)r^3\cos\theta/4\right], \quad 0\Big\}^{\mathrm{T}}.
\end{aligned} \tag{44}$$

Substituting the boundary conditions derives to that the function φ_i ($i = 1, 2$) should be the solution of the Neumann problem,

$$\begin{aligned}
&\nabla^2\varphi_i = 0 \quad (i = 1, 2), \\
&\left.\frac{\mathrm{d}\varphi_1}{\mathrm{d}n}\right|_{\partial\Omega} = -\frac{3+2\nu}{4}r^2\cos\theta l + \frac{1-2\nu}{4}r^2\sin\theta m; \\
&\left.\frac{\mathrm{d}\varphi_2}{\mathrm{d}n}\right|_{\partial\Omega} = -\frac{3+2\nu}{4}r^2\sin\theta l - \frac{1-2\nu}{4}r^2\cos\theta m.
\end{aligned} \tag{45}$$

Because the selection of central principle coordinates, the solutions of eqn (44), must exist, their solution technique can have various approaches but we will not go further here. The solutions of the original problem can be expressed as

$$\psi = \psi_i^{(3)} + z\psi_i^{(2)} + \frac{z^2}{2}\psi_i^{(1)} + \frac{z^3}{6}\psi_i^{(0)} \quad (i = 1, 2). \tag{46}$$

To give the physical meaning, based on the superposition principle, using solutions (40) and (44) gives

$$\psi = \psi_i^{(3)} - L\psi_i^{(2)} + z(\psi_i^{(2)} - L\psi_i^{(1)}) + \frac{z^2}{2}(\psi_i^{(1)} - L\psi_i^{(0)}) + \frac{z^3}{6}\psi_i^{(0)} \quad (i = 1, 2), \tag{47}$$

which are the solutions for clamped at the $z = 0$ end with the $z = L$ end subjected to a force in two directions, i.e., the shearing-bending solutions. Certainly, the end boundary condition is satisfied in the Saint-Venant sense. With the same argument, it can be shown that the fourth order Jordan form subsidiary solution does not exist (Zhong and Xu, 1996). All the conclusions above are valid for arbitrary cross sections for the cylindrical domain.

5. THE NON-ZERO EIGENVALUE SOLUTIONS

All the Saint-Venant solutions have been derived from the eigenvalue zero solutions. However, these solutions are obviously incomplete in the whole solution space. The eigen-equation of the Hamiltonian operator matrix must have other non-zero eigenvalue solutions. In dealing with the two end boundary conditions these solutions are necessary. In the traditional method, these solutions were covered by the well-known Saint-Venant principle and have only local effect near both ends of the cylindrical domain. In this section, the non-zero eigenvalue solutions will be discussed. Consider eqn (18),

$$(\mathbf{H}-\mu\mathbf{I})\psi = 0 \quad (\mu \neq 0). \tag{48}$$

The general solution of eqn (48) can be expressed by the Papkovitch-Neuber type solution (Stephen and Wang, 1992), which is completed, namely,

$$\begin{aligned}
u &= B_r - a_7\frac{\partial}{\partial r}(B_0+rB_r),\\
v &= B_\theta - a_7\frac{\partial}{\partial\theta}(B_0+rB_r),\\
w &= -\mu a_7(B_0+rB_r),\\
p_1 &= \mu Gr\left[B_r - 2a_7\frac{\partial}{\partial r}(B_0+rB_r)\right],\\
p_2 &= \mu Gr\left[B_\theta - 2a_7\frac{\partial}{r\partial\theta}(B_0+rB_r)\right],\\
p_3 &= Ga_1 r\left(\frac{\partial B_r}{\partial r}+\frac{B_r}{r}+\frac{\partial B_\theta}{r\partial\theta}\right) - 2Ga_7\mu^2 r(B_0+rB_r),
\end{aligned} \tag{49}$$

where $a_7 = (\lambda+G)/[2(\lambda+G)]$,

$$L_\mu B_0 = \left(\frac{\partial^2}{\partial r^2}+\frac{\partial}{r\partial r}+\frac{\partial^2}{r^2\partial\theta^2}+\mu^2\right)B_0 = 0, \quad L_\mu(B_r\mathbf{i}_r+B_\theta\mathbf{i}_\theta) = 0. \tag{50}$$

The boundary conditions

$$\begin{aligned}
&l\left[B_r - 2a_7\frac{\partial}{\partial r}(B_0+rB_r)\right] + m\left[B_\theta - 2a_7\frac{\partial}{r\partial\theta}(B_0+rB_r)\right] = 0,\\
&l\left\{a_1\left(\frac{\partial B_r}{\partial r}+\frac{B_r}{r}+\frac{\partial B_\theta}{r\partial_\theta}\right)+2\left[\frac{\partial B_r}{\partial r} - a_7\frac{\partial^2}{\partial r^2}(B_0+rB_r)\right]\right\}+\\
&\quad + m\left\{\frac{\partial B_\theta}{\partial r}-\frac{B_\theta}{r}+\frac{\partial B_r}{r\partial\theta}-2a_7\frac{\partial^2}{r\partial r\partial\theta}(B_0+rB_r)\right\} = 0,\\
&l\left\{\frac{\partial B_\theta}{\partial r}-\frac{B_\theta}{r}+\frac{\partial B_r}{r\partial\theta}-2a_7\frac{\partial^2}{r\partial r\partial\theta}(B_0+rB_r)\right\}+\\
&\quad + m\left\{a_1\left(\frac{\partial B_r}{\partial r}+\frac{B_r}{r}+\frac{\partial B_\theta}{r\partial\theta}\right)+2\left[\frac{\partial B_\theta}{r\partial\theta}+\frac{B_r}{r}-a_7\left(\frac{\partial^2}{r^2\partial\theta^2}+\frac{\partial}{r\partial r}\right)(B_0+rB_r)\right]\right\} = 0.
\end{aligned} \tag{51}$$

In accordance with eqns (25) and (28) and expanding, we write

$$(B_0, B_r, B_\theta) = \sum_n (R_0^{(n)}(r), \quad R_r^{(n)}(r), \quad iR_\theta^{(n)}(r))e^{n\theta i}e^{\mu z}. \tag{52}$$

Substituting eqns (52) into eqns (50), gives

$$\left(\frac{d^2}{dr^2} + \frac{d}{dr} + \mu^2 - \frac{n^2}{r^2}\right)R_0^{(n)} = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{d}{dr} + \mu^2 - \frac{n^2+1}{r^2}\right)R_r^{(n)} + \frac{2n}{r^2}R_\theta^{(n)} = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{d}{dr} + \mu^2 - \frac{n^2+1}{r^2}\right)R_\theta^{(n)} + \frac{2n}{r^2}R_r^{(n)} = 0. \tag{53}$$

The solutions of eqns (53) can be obtained as

$$R_0^{(n)} = C_1^{(n)}J_n(\mu r),$$
$$R_r^{(n)} = C_2^{(n)}J_{n+1}(\mu r) + C_3^{(n)}J_{n-1}(\mu r),$$
$$R_\theta^{(n)} = -C_2^{(n)}J_{n+1}(\mu r) + C_3^{(n)}J_{n-1}(\mu r), \tag{54}$$

where $J_n(*)$ is the Bessel function of order n and $C_k^{(n)}$ $(k = 1, 2, 3)$ are integral constants. The solution (49) of eqn (48) can be rewritten as

$$\psi = \sum_n \psi^{(n)}e^{n\theta i}e^{\mu z}, \tag{55}$$

where the components of $\psi^{(n)}$ are

$$u^{(n)} = C_2^{(n)}J_{n+1} + C_3^{(n)}J_{n-1} - a_7(d/dr)(C_2^{(n)}rJ_{n+1} + C_3^{(n)}rJ_{n-1} + C_1^{(n)}J_n),$$
$$v^{(n)} = i[-C_2^{(n)}J_{n+1} + C_3^{(n)}J_{n-1} - na_7(C_2^{(n)}rJ_{n+1} + C_3^{(n)}rJ_{n-1} + C_1^{(n)}J_n)/r],$$
$$w^{(n)} = -\mu a_7(C_2^{(n)}rJ_{n+1} + C_3^{(n)}rJ_{n-1} + C_1^{(n)}J_n),$$
$$p_1^{(n)} = 2G\mu a_7\{C_2^{(n)}[\mu r^2 J_n + (n+a_8)rJ_{n+1}] - C_3^{(n)}[(n-a_8)rJ_{n-1} + \mu r^2 J_n] + C_1^{(n)}(\mu rJ_{n-1} + nJ_n)\},$$
$$p_2^{(n)} = -2G\mu a_7 i[C_2^{(n)}(n+a_8)rJ_{n+1} + C_3^{(n)}(n-a_8)rJ_{n-1} + C_1^{(n)}nJ_n],$$
$$p_3^{(n)} = \mu a_7 E\{C_2^{(n)}[(a_8-2)rJ_n - \mu r^2 J_{n+1}] - C_3^{(n)}[(a_8-2)rJ_n + \mu r^2 J_{n-1}] - C_1^{(n)}\mu rJ_n\}, \tag{56}$$

where $a_8 = 1/(2a_7)$. The solutions of the cylinder problem should be the linear combination of solution (55) and superimposed zero eigensolutions, which satisfies the end boundary conditions.

For a special case, consider a circular cylinder. Its boundary contour is $r = b$ (b is a constant). Substituting solutions (56) into (52) and (51), gives other forms of the boundary condition

$$[A_{kl}^{(n)}]\mathbf{C}^{(n)} = 0, \tag{57}$$

where the vector $\mathbf{C}^{(n)} = \{C_1^{(n)}, C_2^{(n)}, C_3^{(n)}\}^T$, $A_{11}^{(n)} = -\mu bJ_{n-1} - nJ_n$, $A_{12}^{(n)} = -\mu b^2 J_n - (a_8+n)bJ_{n+1})$, $A_{13}^{(n)} = \mu b^2 J_n - (a_8-n)bJ_{n-1}$, $A_{21}^{(n)} = n(n+1)J_n + n\mu bJ_{n-1}$, $A_{22}^{(n)} = (n+1)(n+2a_8)bJ_{n+1} + (a_8+n)\mu b^2 J_n$, $A_{23}^{(n)} = (n-1)(n+2a_8)bJ_{n-1} + (a_8-n)\mu b^2 J_n$, $A_{31}^{(n)} = (\mu^2 b^2 - n^2 - n)J_n - \mu bJ_{n-2}$, $A_{32}^{(n)} = -(1+a_8)\mu b^2 J_n + [\mu^2 b^2 - (n+1)^2 - (a_8-1)(n+1)]bJ_{n+1}$,

$A_{35}^{(n)} = (1+a_8)\mu b^2 J_n + [\mu^2 b^2 - (n-1)^2 - (a_8-1)(n-1)]nJ_{n-1}$. Eigenvalues μ_n, decay rates, can be obtained by

$$|\mathbf{A}_{kl}^{(n)}| = 0. \tag{58}$$

The integral constants, $\mathbf{C}^{(n)}$, can be determined by the end boundary conditions. The form of the solution is the same as Stephen and Wang (1992).

6. CONCLUSION

The solution method for elasticity is a very important part of mathematical physics. The semi-inverse solution method of the Saint-Venant is still fundamental in solving elasticity problems. Most books on elasticity published so far, for the solution technique of the Saint-Venant problem, still use the semi-inverse method. After the introduction of the Hamiltonian system expression, classical solutions of the Saint-Venant are found to be solved by the direct method, which is in the symplectic space under the Hamiltonian system formulation but not in the Euclidian space under the traditional Lagrange system formulation.

The traditional semi-inverse solution method of the Saint-Venant problem can only find some solutions, but cannot assert if there have been any further solutions, nor how to find the remaining solutions. In the direct method, all the Saint-Venant solutions have been derived from the zero eigenvalue solutions and the others do not exist. However, these solutions are obviously incomplete in the whole solution space. The eigenequation of the Hamiltonian operator matrix must have other non-zero eigenvalue solutions. In dealing with the two end boundary condition, those solutions, which were covered by the well-known Saint-Venant principle, are obtained from non-zero eigenvalue problems. These solutions have only local effects. The Saint-Venant problem and the Saint-Venant principle have been unified by the method. Direct solution methods should be strongly demanded. It is now the time to rescan various fields in applied mechanics based on the view point of Hamiltonian system formulation, that the system and geometry updating will bring fruitful opportunities for further researches.

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